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HARMONIC WAVES IN ONE-, TWO- AND THREE-DIMENSIONAL COMPOSITES: BOUNDS FOR EIGENFREQUENCIES†

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Abstract—Harmonic waves in one-, two- and three-dimensional elastic composites with periodic structure are considered. Based on a new quotient recently proposed by Nemat-Nasser, lower and upper bounds for the eigenfrequencies are developed. For illustration waves propagating normal to the layers in layered composites, and normal to the fibers in fiber-reinforced composites, are considered. These examples show that the new quotient is very effective and yields very accurate results for the considered class of problems. While these results are of interest in their own right, they can be used to check the effectiveness of various approximate theories which recently have been proposed by various authors.

1. INTRODUCTION

This paper is concerned with eigenfrequencies of harmonic waves in one-, two- and three-dimensional periodic elastic composites. The approach is based on a general variational method in which the stress and displacement are varied independently [1–3]. Since the elasticity tensor in elastic composites admits large relative discontinuities, it turns out that the usual Rayleigh quotient which in conjunction with a Rayleigh–Ritz method provides upper bounds for the eigenfrequencies, is not quite effective, whereas a new quotient based on the more general variational method yields extremely accurate results. When the mass-density is constant, and the test (or the approximating) functions are suitable (for example the usual Fourier exponential functions), then the new quotient provides upper bounds. Also, when the elasticity tensor is constant, and exponential Fourier functions are used, the new quotient reduces to the Rayleigh quotient, and hence gives upper bounds. In general, however, the results of the new quotient are neither lower nor upper bounds for the exact eigenfrequencies. Based on the new quotient, the present paper gives both lower and upper bounds, and discusses their effectiveness. With the stress as the only unknown field, an upper bound is developed, which turns out to be much better than the corresponding bound given by the usual Rayleigh quotient, for cases in which the elasticity tensor admits large variations, while the mass-density varies slowly.

In Section 2 the considered problem is defined, and the corresponding Rayleigh quotient is stated and compared with the new quotient. In Section 3 some relevant properties of the exact eigenfuctions and eigenfrequencies are examined. The bounds are discussed in Section 4, and the approximate nature of the new quotient is examined in Section 5. For illustration, the problem of harmonic waves propagating normal to the layers in layered composites, and normal to the fibers in fiber-reinforced composites, are treated in Section 6. Since for two- and three-dimensional

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composites no exact solution exists, the results should also prove useful in checking the accuracy and effectiveness of various approximate theories which have been proposed recently [4–8] by various authors for the treatment of waves in elastic composites.

In passing it should be noted that there are various methods by which lower and upper bounds of eigenvalues of elliptic operators can be obtained; for discussion and further references see [9]. The present method relates to the one which finds its root in the work of Krylov and Bogoliubov [10], Weinstein [11], Kohn [12] and Kato [13]. The mathematical procedure in the present work, however, is more complicated, because of the more general variational method used, although the final results are quite simple.

2. STATEMENT OF PROBLEM

Consider harmonic waves in an unbounded elastic composite consisting of a collection of completely bonded, identical unit cells which repeat themselves in all directions, and hence form a periodic structure. A unit cell may consist of different constituents with differing properties, and may have various geometries similar to those occurring in space lattices of crystals. For example, in a layered composite, a unit cell may consist of a finite number of layers of elastic materials, each layer being (possibly) heterogeneous along its thickness. In a fiber-reinforced composite, on the other hand, a unit cell is an infinitely extended cylinder with, say, rectangular or hexagonal cross-sections. Within each cell there may be a number of cylindrical fibers of various cross-sections, whose axes are parallel to the generator of the cylindrical cell. Figure 1 shows a cell with rectangular cross-section containing one rectangular fiber, whereas the fiber in Fig. 2 has an elliptical cross-section. In a three-dimensional structure, these fibers may have a common length l, and may be spaced at an equal distance in the direction perpendicular to the plane of Figs. 1 and 2. The present theory applies to these and similar one-, two- and three-dimensional composites of periodic structures.

For the sake of simplicity in presentation, assume that a unit cell is in the form of a parallelepiped whose three edges are defined by the three vectors $\mathbf{l}^{\beta}$, $\beta = 1, 2, 3$. Denote the region occupied by this unit cell by $\mathfrak{R}$, having boundary $\partial\mathfrak{R}$, and let Σ be the collection of all

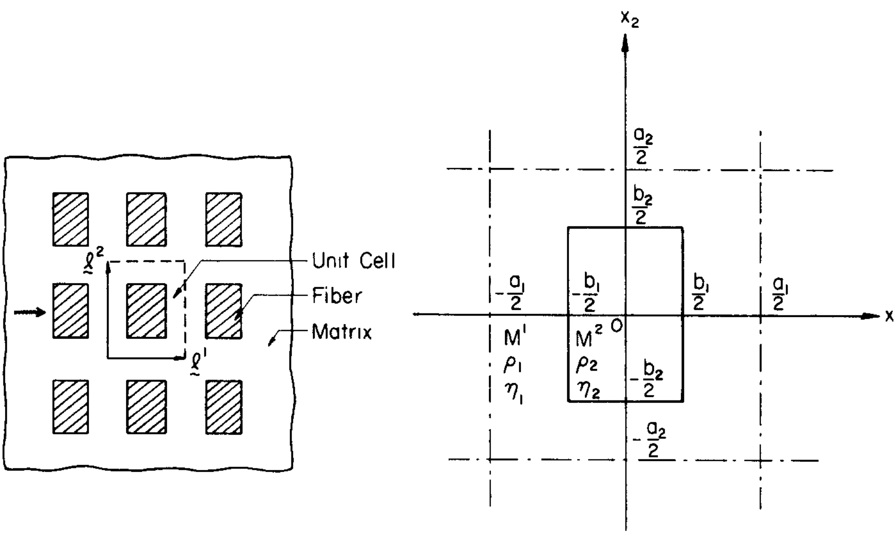

Fig. 1.

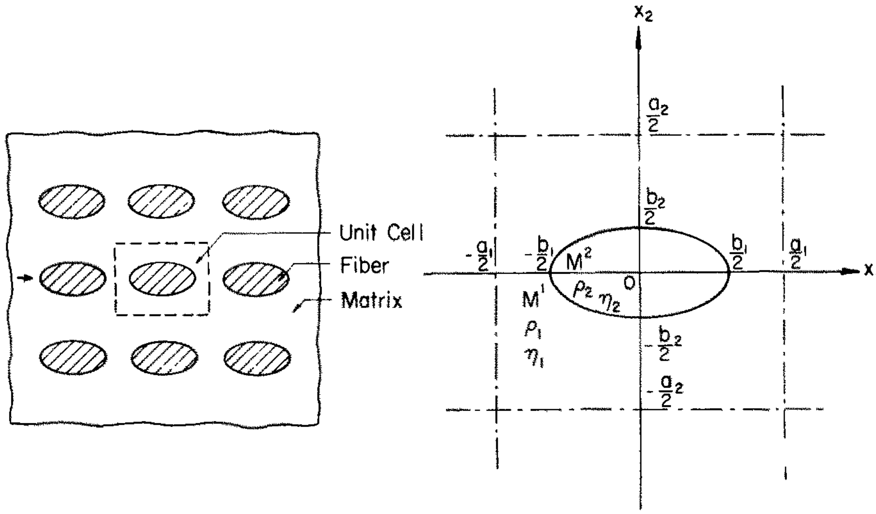

Fig. 2.

interior surfaces which separate different material constituents within the cell. It will be assumed that Σ has a continuously turning tangent plane, although the results are immediately applicable to cases in which Σ has sharp edges and corners that are formed by the intersection of two or three smooth surfaces. In these latter cases, the displacement-gradient may become unbounded at these corners and edges, while still remaining square integrable, so that the total strain-energy within a cell is finite; the displacement field, of course, is always continuous and bounded everywhere in $\mathcal{R}$.

The mass-density $\rho(\mathbf{x})$ and the elasticity tensor with rectangular Cartesian components† $C_{jkmn}(\mathbf{x})$ are continuous and continuously differentiable functions in the subregion occupied by each constituent, but in general suffer finite discontinuities across Σ; here $\mathbf{x}$ is the position vector with components x_j, $j = 1, 2, 3$. It will be assumed that along the unit normal $\mathbf{n}$ on Σ, these functions admit right- and left-hand limits and derivatives; they are piece-wise continuously differentiable in $\mathcal{R}$. Observe that, in view of the periodicity of the composite, one has $\rho(\mathbf{x}) = \rho(\mathbf{x} + m'\mathbf{l}^\beta)$, $C_{jkmn}(\mathbf{x}) = C_{jkmn}(\mathbf{x} + m'\mathbf{l}^\beta)$, where m' is an integer.

In general, C_{jkmn} is symmetric with respect to the exchange of j and k, m and n, and jk and mn. It is, moreover, positive-definite in the sense that, at every $\mathbf{x}$ in $\mathcal{R}$, and for every real-valued *nonzero* symmetric tensor ξ_{ij}, there exists a positive $\mu(\mathbf{x})$ such that‡

$$C_{jkmn}\xi_{jm}\xi_{kn} \geq \mu(\mathbf{x})\xi_{jm}\xi_{mj} > 0. \tag{2.1}$$

For harmonic waves with frequency ω, all the field quantities are proportional to $e^{\pm i\omega t}$, where $i = \sqrt{-1}$ and t measures time. Thus the field equations become

$$\sigma_{jk,k} + \lambda\rho u_j = 0, \qquad \sigma_{jk} = C_{jkmn}u_{m,n}, \qquad j, k, m, n = 1, 2, 3, \tag{2.2}$$

where $\lambda = \omega^2$; a comma followed by an index denotes differentiation with respect to the

†Throughout this paper all quantities will be referred to a fixed rectangular Cartesian coordinate system.

‡In application, the elasticity tensor is usually piece-wise constant, and in most cases, isotropic, having the form $C_{jkmn} = \lambda\delta_{jk}\delta_{mn} + (\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{mk})\mu$, where λ and μ are the Lamé constants, and δ_{jk} is the Kronecker delta.

corresponding coordinate; $\sigma_{jk}e^{\pm i\omega t}$ and $u_j e^{\pm i\omega t}$ are the stress and displacement fields, respectively.†

For harmonic waves with wave vector $\mathbf{q}$, the boundary conditions take on the following quasi-periodic form:

$$u_j(\mathbf{x}+\mathbf{l}^\beta) = u_j(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{l}^\beta},$$
$$t_j(\mathbf{x}+\mathbf{l}^\beta) = -t_j(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{l}^\beta}, \quad \text{for } \mathbf{x} \text{ on } \partial\mathcal{R}, \tag{2.3}$$

where $t_j = \sigma_{jk}n_k$ is the traction vector which must remain continuous across any interior surface. In particular, one must have

$$[\sigma_{jk}(\mathbf{x}^+) - \sigma_{jk}(\mathbf{x}^-)]n_k = 0 \quad \text{for } \mathbf{x} \text{ on } \Sigma, \tag{2.4}$$

where n_k is the unit vector on Σ, pointing from one subregion, say, subregion 1, to the adjacent one, say, subregion 2, $\sigma_{jk}(\mathbf{x}^+)$ is the limiting value of the stress as a point on Σ is approached along the normal from within subregion 1, and $\sigma_{jk}(\mathbf{x}^-)$ is the limit when this point is approached from within subregion 2.

The objective of the analysis is to solve equations (2.2), subject to boundary conditions (2.3) and continuity conditions (2.4), for the continuous and piece-wise continuously differentiable displacement u_j, and to obtain the eigenfrequencies $\omega = \sqrt{\lambda}$ as functions of the wave vector $\mathbf{q}$. Note that, for nonzero $\mathbf{q}$, conditions $(2.3)_1$ preclude nonzero constant solutions.

It will be assumed that the eigenvalue problem

$$-\frac{1}{\rho}(C_{jkmn}\,u_{m,n})_{,k} = \lambda\, u_j \tag{2.5}$$

with boundary conditions (2.3), σ_{jk} being defined by $(2.2)_2$, admits an infinite set of positive distinct eigenvalues $0<\lambda_1<\lambda_2<\lambda_3\ldots<\lambda_n\ldots$ which become unbounded with n. The corresponding orthonormal (with weighting function ρ) eigenvectors will be denoted by $\{\varphi_j^{(n)}\}$, so that

$$\langle\rho\varphi_j^{(n)}, \varphi_j^{(m)}\rangle = \delta_{mn}, \tag{2.6}$$

where δ_{mn} is the Kronecker delta, and the inner product is defined by

$$\langle g u_j, v_j\rangle \equiv \int_{\mathcal{R}} g u_j v_j^* \,\mathrm{d}V, \tag{2.7}$$

g being a suitable real-valued weighting function, and * denoting the complex conjugate.

It is known that the pth eigenvalue is given by the infimum of the Rayleigh quotient‡

$$\bar{\lambda}_R = \langle C_{jkmn}u_{m,n}, u_{j,k}\rangle/\langle\rho u_j, u_j\rangle, \tag{2.8}$$

†In the sequel σ_{jk} and u_j will be referred to as the stress and displacement fields.

‡Quotient (2.8) will be called the "displacement Rayleigh quotient" as contrasted with (4.7) which is the "stress Rayleigh quotient".

subject to the constraints

$$\langle \rho u_j, \varphi_j^{(n)} \rangle = 0, \qquad n = 1, 2, \ldots, p-1, \tag{2.9}$$

where this infimum is taken on the set of all vector-valued (with complex-valued components) functions, **V**, which, say, is continuous and piece-wise continuously differentiable, and which satisfies $(2.3)_1$. If the class of functions which satisfy $(2.3)_1$, and for which the integrals in the right-hand side of (2.8) are finite, is denoted by $\bar{\mathbf{V}}$, then the pth eigenvalue is given by the minimum of (2.8) on this class, subject to constraints (2.9). Denote by $\mathbf{V}'$ the subset of $\bar{\mathbf{V}}$, which also satisfies (2.4).

In practice it is very difficult to choose test functions which satisfy continuity conditions (2.4). One often selects an orthogonal† sequence of functions $\{f^{(\alpha\beta\gamma)}(\mathbf{x})\}$, $\alpha, \beta, \gamma = 1, 2, \ldots, M$, which are continuous and continuously differentiable, and which satisfy the quasi-periodicity condition $(2.3)_1$. Then, one considers an *approximate* solution

$$\bar{u}_j = \sum_{\alpha,\beta,\gamma=1}^{M} U_j^{(\alpha\beta\gamma)} f^{(\alpha\beta\gamma)}(\mathbf{x}), \tag{2.10}$$

substitutes into (2.8), and minimizes $\bar{\lambda}_R$ with respect to the unknown coefficients $U_j^{(\alpha\beta\gamma)}$, to arrive at a set of $3M^3$ linear homogeneous equations for these coefficients.‡ The roots of the determinant of the coefficients in these equations then yield upper bounds for the corresponding first $3M^3$ eigenfrequencies.§

For composites in which the elasticity tensor admits large discontinuities, this method is not effective. To remedy this, the following method[1–3] which turns out to be extremely effective, can be used.

Instead of (2.5), consider the original equations (2.2), and let $\bar{\boldsymbol{v}}$ be the set of all second-order symmetric tensor-valued functions with complex-valued components σ_{jk} which are each square integrable in $\mathcal{R}$. Assume further that $\bar{\boldsymbol{v}}$ satisfies $(2.3)_2$. Denote by $\boldsymbol{v}'$ the subset of $\bar{\boldsymbol{v}}$, which is piece-wise continuous and continuously differentiable in $\mathcal{R}$, and hence satisfies the continuity conditions (2.4). Now, the elements u_j of **V** and σ_{jk} of $\boldsymbol{v}'$, which satisfy (2.2), render the following functional stationary, as can be verified by direct calculation:

$$\lambda_N = (\langle \sigma_{jk}, u_{j,k} \rangle + \langle u_{j,k}, \sigma_{jk} \rangle - \langle D_{jkmn}\sigma_{jk}, \sigma_{mn} \rangle)/\langle \rho u_j, u_j \rangle, \tag{2.11}$$

where D_{jkmn} is the elastic compliance¶ whose matrix is the inverse of that of C_{jkmn}; note that D_{jkmn} is positive-definite in the sense of (2.1).

For *approximate* solutions set

†It is not necessary that this sequence of functions be orthogonal, but they must be linearly independent. This orthogonality is defined such that $\langle f^{(\alpha\beta\gamma)}, f^{(\theta\eta\xi)} \rangle$ is proportional to $\delta_{\alpha\theta}\, \delta_{\beta\eta}\, \delta_{\gamma\xi}$, $\delta_{\alpha\theta}$ being the M-dimensional Kronecker delta.

‡Note that the superposed bar in (2.10) denotes the approximate solution.

§Collaz[14], p. 239, calls (2.8) Kamke quotient, for which the test functions need only satisfy the "essential" boundary conditions $(2.3)_1$; conditions $(2.3)_2$ and (2.4) can be suppressed. With suitable test functions, the Rayleigh–Ritz method always gives upper bounds. This follows from Poincare's minimum-maximum principle.

¶For isotropic materials and in plane strain, for example, this is given by

$$D_{jkmn} = \frac{1}{2\mu}\left[\frac{1}{2}(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}) - \frac{\lambda}{2(\mu+\lambda)}\delta_{jk}\delta_{mn}\right],$$

where λ and μ are Lamé coefficients, and δ_{jk} is the two-dimensional Kronecker delta.

$$\bar{\sigma}_{jk} = \sum_{\alpha,\beta,\gamma=1}^{M} S_{jk}^{(\alpha\beta\gamma)} f^{(\alpha\beta\gamma)}(\mathbf{x}), \tag{2.12}$$

substitute this and (2.10) into (2.11), and equate to zero the derivatives of λ_N with respect to the unknown coefficients, to arrive at the following set of $9M^3$ linear homogeneous equations for these unknowns:†

$$\begin{aligned} &\langle \bar{\sigma}_{jk,k} + \lambda_N \rho \bar{u}_j, f^{(\alpha\beta\gamma)} \rangle = 0, \\ &\langle D_{jkmn}\bar{\sigma}_{mn} - \bar{u}_{(j,k)}, f^{(\alpha\beta\gamma)} \rangle = 0, \qquad j, k, m, n = 1, 2, 3, \\ &\qquad \alpha, \beta, \gamma = 1, 2, \ldots, M. \end{aligned} \tag{2.13}$$

Equations $(2.13)_2$, which are $6\,M^3$ in number, can be solved for $S_{jk}^{(\alpha\beta\gamma)}$ in terms of $U_j^{(\alpha\beta\gamma)}$, and the results substituted into $(2.13)_1$. This leads to a system of $3M^3$ linear homogeneous equations for the latter unknowns. The roots of the determinant of the coefficients of these unknowns then are the approximate values of the first $3M^3$ eigenfrequencies. The corresponding approximate eigenfunctions are then given by (2.10) and (2.12), where (2.10) can be normalized as

$$\langle \rho \bar{u}_j, \bar{u}_j \rangle = 1 \tag{2.14}$$

which also normalizes (2.12).

Denote by $\lambda_N^{(p)}$ the approximate pth eigenvalue obtained from the new quotient in the manner discussed above, and note from the structure of equation (2.11) that this eigenvalue is always real. Designate the corresponding displacement and stress fields by $\bar{u}_j^{(p)}$ and $\bar{\sigma}_{jk}^{(p)}$, respectively, where $p = 1, 2, \ldots, 3M^3 = \bar{M}$.

Remark 1. Subject to the normalization (2.14), the approximate displacement and stress fields are each an orthogonal set in the following sense:

$$\begin{aligned} &\langle \rho \bar{u}_j^{(p)}, \bar{u}_j^{(q)} \rangle = \delta_{pq}, \\ &\langle D_{jkmn}\bar{\sigma}_{jk}^{(p)}, \bar{\sigma}_{mn}^{(q)} \rangle = \sqrt{\lambda_N^{(p)}\lambda_N^{(q)}}\,\delta_{pq}. \end{aligned} \tag{2.15}$$

Proof. From (2.13) and the fact that $\bar{\sigma}_{jk}^{(p)} = \bar{\sigma}_{kj}^{(p)}$ one first obtains

$$\begin{aligned} \langle D_{jkmn}\bar{\sigma}_{jk}^{(p)}, \bar{\sigma}_{mn}^{(q)} \rangle &= \langle \bar{u}_{j,k}^{(p)}, \bar{\sigma}_{jk}^{(q)} \rangle \\ &= -\langle \bar{u}_j^{(p)}, \bar{\sigma}_{jk,k}^{(q)} \rangle, \\ \langle \bar{u}_j^{(p)}, \bar{\sigma}_{jk,k}^{(q)} \rangle &= -\lambda_N^{(q)} \langle \rho \bar{u}_j^{(p)}, \bar{u}_j^{(q)} \rangle. \end{aligned} \tag{2.16}$$

From the combination of $(2.16)_{1,2}$, it now follows that

$$\langle D_{jkmn}\bar{\sigma}_{jk}^{(p)}, \bar{\sigma}_{mn}^{(q)} \rangle = \lambda_N^{(q)} \langle \rho \bar{u}_j^{(p)}, \bar{u}_j^{(q)} \rangle. \tag{2.17}$$

Exchanging the role of p and q, and using the resulting equation with (2.17) and (2.16), one immediately arrives at equations (2.15).

Henceforth it will be assumed that all the displacement fields are normalized according to (2·14).

†Note that, because of the symmetry of σ_{jk}, $S_{jk}^{(\alpha\beta\gamma)} = S_{kj}^{(\alpha\beta\gamma)}$. Also, note that $u_{(j,k)} = \frac{1}{2}[u_{j,k} + u_{k,j}]$.

Remark 2. Set $p = q$ in $(2.15)_2$ and obtain

$$\langle D_{jkmn}\bar{\sigma}_{jk}^{(p)}, \bar{\sigma}_{mn}^{(p)}\rangle = \lambda_N^{(p)}. \tag{2.18}$$

Whereas in the case of the Rayleigh quotient (2.8) one arrives, by a Rayleigh–Ritz method, at upper bounds for the eigenfrequencies, the new quotient (2.11) in general yields neither upper nor lower bounds. It is therefore interesting to note the relation between $\bar{\lambda}_R$ and λ_N as follows.

Remark 3. For every element u_j in $\bar{\mathbf{V}}$ and σ_{jk} in $\bar{\mathbf{v}}$, one has

$$\bar{\lambda}_R^{(1)} \geq \lambda_N^{(1)} \tag{2.19}$$

Proof. Calculate $\bar{\lambda}_R^{(1)} - \lambda_N^{(1)}$ as follows:

$$\bar{\lambda}_R^{(1)} - \lambda_N^{(1)} = \langle C_{jklm}u_{j,k}, u_{l,m}\rangle + \langle D_{jklm}\sigma_{jk}, \sigma_{lm}\rangle - \langle \sigma_{jk}, u_{j,k}\rangle - \langle u_{j,k}, \sigma_{jk}\rangle$$

$$= \{\langle C_{jkmn}(u_{j,k} - D_{jkrs}\sigma_{rs}), (u_{m,n} - D_{mnpq}\sigma_{pq})\rangle\} \geq 0. \tag{2.20}$$

Note, however, that, although $\bar{\lambda}_R^{(1)}$ is not less than the exact eigenvalue, and that $\lambda_N^{(1)}$ is not greater than $\bar{\lambda}_R^{(1)}$, one cannot in general conclude that $\lambda_N^{(1)}$ should necessarily be better than $\bar{\lambda}_R^{(1)}$, although all examples considered so far strongly suggest that this indeed may be the case. When the mass-density is constant and the approximating functions are suitable, for example they are in the form

$$f^{(\alpha\beta\gamma)} = e^{i[\mathbf{q}\cdot\mathbf{x}+2\pi(\alpha x_1+\beta x_2+\gamma x_3)]}, \tag{2.21}$$

then, as is shown in Section 4, $\lambda_N^{(p)}$ becomes an upper bound for the exact λ_p. In Section 6 it is illustrated that $\lambda_N^{(p)}$ is then a very accurate upper bound for the considered class of problems.

3. ON PROPERTIES OF EIGENFUNCTIONS AND EIGENFREQUENCIES

On the space $\bar{\mathbf{V}}$ with inner product $\langle\rho u_j, u_j\rangle$, the orthonormal set of eigenvectors $\{\varphi_j^{(n)}\}$ is complete. Since the approximate solution $\bar{u}_j^{(p)}$ is continuous and continuously differentiable in $\mathcal{R}$, it has the Fourier series

$$\bar{u}_j^{(p)} = \sum_{n=1}^{\infty} C_n^{(p)}\varphi_j^{(n)}, \quad C_n^{(p)} = \langle\rho\bar{u}_j^{(p)}, \varphi_j^{(n)}\rangle, \quad j = 1, 2, 3, \tag{3.1}$$

where the coefficients $(3.1)_2$ satisfy Parseval's equality

$$\sum_{n=1}^{\infty} |C_n^{(p)}|^2 = \langle\rho u_j^{(p)}, u_j^{(p)}\rangle = 1, \text{(no sum on } p).† \tag{3.2}$$

It will be useful to set

$$\psi_{jk}^{(n)} = C_{jklm}\varphi_{l,m}^{(n)}, \qquad j, k, l, m = 1, 2, 3, \tag{3.3}$$

†Unless otherwise stated explicitly, only repeated subscripts (but not superscripts) are to be summed over their range of variation.

rewrite $(2.2)_1$ as

$$\psi^{(n)}_{jk,k} + \lambda_n \rho \varphi_j^{(n)} = 0, \tag{3.4}$$

and observe that $\psi^{(n)}_{jk}$ satisfies

$$(R\psi^{(n)}_{jk,k})_{,l} + \lambda_n D_{jlrs}\psi^{(n)}_{rs} = 0, \quad j, k, l, r, s = 1, 2, 3, \tag{3.5}$$

where $R = 1/\rho$. It now immediately follows that the sequences of vector-valued and symmetric tensor-valued functions, $\{\psi^{(n)}_{jk,k}/\lambda_n\}$ and $\{\psi^{(n)}_{jk}/\sqrt{\lambda_n}\}$, are orthonormal in the following sense:

$$\langle R\psi^{(n)}_{jk,k}/\lambda_n, \psi^{(m)}_{jl,l}/\lambda_m\rangle = \delta_{nm},$$
$$\langle D_{jkrs}\psi^{(n)}_{jk}/\sqrt{\lambda_n}, \psi^{(m)}_{rs}/\sqrt{\lambda_m}\rangle = \delta_{nm}, \text{(no sum on } n \text{ and } m).† \tag{3.6}$$

The form $(3.6)_1$ defines a norm for the space $\bar{\mathbf{V}}$, whereas the form $(3.6)_2$ gives a norm on $\bar{\mathfrak{v}}$, so that for every u_j in $\bar{\mathbf{V}}$ and every $\sigma_{jk} = \sigma_{kj}$ in $\bar{\mathfrak{v}}$, one has

$$\|u_j\|^2 = \langle \rho u_j, u_j\rangle, \|\sigma_{jk}\|^2 = \langle D_{jkrs}\sigma_{jk}, \sigma_{rs}\rangle. \tag{3.7}$$

From (3.4) and the completeness of $\{\varphi_j^{(n)}\}$, it follows that $\{\psi^{(n)}_{jk,k}/\lambda_n\}$ is also complete on $\bar{\mathbf{V}}$.

For the approximate solution $\bar{\sigma}^{(p)}_{jk}$ which is continuous and continuously differentiable in $\mathcal{R}$, consider the following formal Fourier series:

$$\bar{\sigma}^{(p)}_{jk} \sim \sum_{n=1}^{\infty} A_n^{(p)}\psi^{(n)}_{jk}, \qquad \bar{\sigma}^{(p)}_{jk,k} \sim \sum_{n=1}^{\infty} \bar{A}_n^{(p)}\psi^{(n)}_{jk,k}, \tag{3.8}$$

and observe that

$$A_n^{(p)} = \bar{A}_n^{(p)} = \frac{\langle D_{jklm}\bar{\sigma}^{(p)}_{jk}, \psi^{(n)}_{lm}\rangle}{\langle D_{jklm}\psi^{(n)}_{jk}, \psi^{(n)}_{lm}\rangle} = \frac{\langle R\bar{\sigma}^{(p)}_{jk,k}, \psi^{(n)}_{jl,l}\rangle}{\langle R\psi^{(n)}_{jk,k}, \psi^{(n)}_{jl,l}\rangle}. \tag{3.9}$$

From the completeness of $\{\psi^{(n)}_{jk,k}/\lambda_n\}$ it follows that the series in $(3{\cdot}8)_2$ converges in the mean-square sense (with respect to the weighting function R), and one has the Parseval equality for the pth mode,

$$G^{(p)} = \sum_{n=1}^{\infty} \lambda_n^2 |A_n^{(p)}|^2 = \langle R\bar{\sigma}^{(p)}_{jk,k}, \bar{\sigma}^{(p)}_{jl,l}\rangle. \tag{3.10}$$

Remark 4. On the complete space $\bar{\mathfrak{v}}$ whose norm is defined by $(3.7)_2$, the orthonormal set $\{\psi^{(n)}_{jk}/\sqrt{\lambda_n}\}$ is complete.

The proof of this assertion may be sketched as follows. From (3.5) it follows that

$$\lambda_n = \langle R\psi^{(n)}_{jk,k}, \psi^{(n)}_{jl,l}\rangle / \langle D_{jklm}\psi^{(n)}_{jk}, \psi^{(n)}_{lm}\rangle \tag{3.11}$$

†Unless otherwise stated explicitly, only repeated subscripts (but not superscripts) are to be summed over their range of variation.

is the eigenvalue of (2.8), which goes to infinity with n. Suppose now that $\tau_{jk} = \tau_{kj}$, an element of $\boldsymbol{v}$ which is a dense subset of $\bar{v}$, is orthogonal to $\psi_{jk}^{(n)}/\sqrt{\lambda_n}$ for all n. From the Rayleigh quotient it follows that

$$\frac{1}{\lambda_n} \geq \{\langle D_{jklm}\tau_{jk}, \tau_{lm}\rangle / \langle R\tau_{jk,k}, \tau_{jl,l}\rangle\} > 0.$$

But, since $1/\lambda_n$ goes to zero with n, one obtains $\langle D_{jklm}\tau_{jk}, \tau_{lm}\rangle = 0$. However, the left-hand side of this equation is positive-definite (see (2.1)), which implies that $\tau_{jk} = 0$.

From this result one observes that the series in $(3.8)_1$ converges with respect to the norm $(3.7)_2$. One also has the following Parseval's equality:

$$\sum_{n=1}^{\infty} \lambda_n |A_n^{(p)}|^2 = \langle D_{jklm}\bar{\sigma}_{jk}^{(p)}, \bar{\sigma}_{lm}^{(p)}\rangle = \|\bar{\sigma}_{jk}^{(p)}\|^2. \tag{3.12}$$

Comparison with (2.15) now readily shows that

$$\lambda_N^{(p)} = \langle \bar{\sigma}_{jk}^{(p)}, u_{j,k}^{(p)}\rangle = \langle D_{jklm}\bar{\sigma}_{jk}^{(p)}, \bar{\sigma}_{lm}^{(p)}\rangle$$
$$= \sum_{n=1}^{\infty} \lambda_n |A_n^{(p)}|^2. \tag{3.13}$$

This and (3.10) will be useful in the following developments.

4. UPPER AND LOWER BOUNDS FOR EIGENFREQUENCIES

Since the *exact* eigenfrequencies are discrete, positive, and ordered in the ascending manner, for positive fixed integers n and p, one has $(\lambda_n - \lambda_p)(\lambda_n - \lambda_{p+1}) \geq 0$, which yields

$$\lambda_n^2 - (\lambda_p + \lambda_{p+1})\lambda_n + \lambda_p\lambda_{p+1} \geq 0. \tag{4.1}$$

Consider now the approximate pth eigenvalue calculated from (2.11) and (2.13), and denote it and its corresponding approximate eigenfunctions by $\lambda_N^{(p)}$, $\bar{u}_j^{(p)}$ and $\bar{\sigma}_{jk}^{(p)}$. With (3.10) and (3.13) in mind, transform (4.1) to†

$$\sum_{n=1}^{\infty} \lambda_n^2 |A_n^{(p)}|^2 - (\lambda_p + \lambda_{p+1}) \sum_{n=1}^{\infty} \lambda_n |A_n^{(p)}|^2 + \lambda_p\lambda_{p+1} \sum_{n=1}^{\infty} |A_n^{(p)}|^2 \geq 0. \tag{4.2}$$

Now consider the approximate $\lambda_N^{(p)}$, and define

$$\bar{\lambda}_N^{(p)} = \lambda_N^{(p)}/K^{(p)}, \qquad K^{(p)} = \sum_{n=1}^{\infty} |A_n^{(p)}|^2. \tag{4.3}$$

Remark 5. If $\bar{\lambda}_N^{(p)}$ is such that $\lambda_{p-1} < \bar{\lambda}_N^{(p)} < \lambda_{p+1}$, one has the following bounds for the *exact* pth eigenvalue λ_p:

$$\bar{\lambda}_N^{(p)} - \frac{\bar{G}^{(p)} - (\bar{\lambda}_N^{(p)})^2}{\lambda_{p+1} - \bar{\lambda}_N^{(p)}} \leq \lambda_p \leq \bar{\lambda}_N^{(p)} + \frac{\bar{G}^{(p)} - (\bar{\lambda}_N^{(p)})^2}{\bar{\lambda}_N^{(p)} - \lambda_{p-1}}, \tag{4.4}$$

†In all equations in this and the remaining sections, p is fixed and hence no sum is implied on repeated subscript or superscript p's.

where

$$\bar{G}^{(p)} = \langle R\bar{\sigma}^{(p)}_{jk,k}, \bar{\sigma}^{(p)}_{jl,l}\rangle / K^{(p)}. \tag{4.5}$$

To show this, from equations (3.10) and (3.13), and definitions (4.3), observe that (4.2) can be rewritten as

$$\bar{G}^{(p)} - (\bar{\lambda}_N{}^{(p)})^2 + (\lambda_p - \bar{\lambda}_N{}^{(p)})(\lambda_{p+1} - \bar{\lambda}_N{}^{(p)}) \geq 0. \tag{4.6}$$

Since λ_{p+1} is greater than $\bar{\lambda}_N{}^{(p)}$, this inequality immediately yields the first part of (4.4). The second part of (4.4) can be obtained in a similar way.

The result (4.4) is only of a formal interest, since it involves quantity $K^{(p)}$, given by equation $(4.3)_2$, which is not known; $K^{(p)}$ can be calculated for the one-dimensional case, as is shown later on. For the first eigenvalue, however, one can obtain another upper bound which is useful, and is sharper than that provided by (4.4),† for cases in which ρ varies slowly and smoothly while C_{jklm} does not.

Upper bound. Of course, the Rayleigh quotient (2.8), in conjunction with a suitable Rayleigh–Ritz method, always gives upper bounds. But for the present class of problems‡ this quotient turns out to give very poor upper bounds. On the other hand, the "stress Rayleigh quotient" corresponding to (3.5), i.e.

$$\bar{\bar{\lambda}}_R = \frac{\langle R\sigma_{jk,k}, \sigma_{jk,l}\rangle}{\langle D_{jlrs}\sigma_{jl}, \sigma_{rs}\rangle}, \tag{4.7}$$

yields, *for the approximate solution* $\bar{\sigma}^{(1)}_{jk}$,

$$\lambda_1 \leq \bar{\bar{\lambda}}_R{}^{(1)}(\bar{\sigma}^{(1)}_{jk}) = \frac{\langle R\bar{\sigma}^{(1)}_{jk,k}, \bar{\sigma}^{(1)}_{rs,s}\rangle}{\langle D_{jlrs}\bar{\sigma}^{(1)}_{jl}, \bar{\sigma}^{(1)}_{rs}\rangle}$$
$$= \frac{\sum_{n=1}^{\infty} \lambda_n{}^2 |A_n{}^{(1)}|^2}{\sum_{n=1}^{\infty} \lambda_n |A_n{}^{(1)}|^2} \tag{4.8}$$

which can be calculated, and happens to yield a much better bound (in particular when R is smooth) than the "displacement Rayleigh quotient" (2.8) which gives a sharper bound when C_{jkmn} is smooth but ρ admits sharp variations or large relative discontinuities; this is illustrated in Section 6 (Table 5).

As of now, it is not clear how the quantity $K^{(p)}$ in (4.3) to (4.5), can be calculated for two- and three-dimensional problems. For the one-dimensional case, however, one can calculate $K^{(p)}$ in terms of the approximate expression for the stress. This is discussed below.

One-dimensional case. In this case, equations (2.2) become

$$\sigma' + \lambda\rho u = 0, \qquad \sigma = \eta u', \qquad -\frac{a}{2} \leq x \leq \frac{a}{2}, \tag{4.9}$$

†In the one-dimensional case where (4.4) can be used directly (see section 6).

‡I.e. when C_{jkmn} has sharp variations within the unit cell, or when it admits large relative discontinuities.

with boundary conditions

$$u\left(\frac{a}{2}\right)=u\left(-\frac{a}{2}\right)e^{iqa},\quad \sigma\left(\frac{a}{2}\right)=\sigma\left(-\frac{a}{2}\right)e^{iqa},$$

where prime denotes differentiation with respect to x, η is the elasticity coefficient, the length of the unit cell is denoted by a, and q is the wave number. The new quotient becomes

$$\lambda_N=(\langle\sigma,u'\rangle+\langle u',\sigma\rangle-\langle D\sigma,\sigma\rangle)/\langle\rho u,u\rangle, \tag{4.10}$$

where $D=1/\eta$. The approximate solutions are

$$\bar{u}=\sum_{\alpha=1}^{M}U^{(\alpha)}f^{(\alpha)},\qquad \bar{\sigma}=\sum_{\alpha=1}^{M}S^{(\alpha)}f^{(\alpha)}, \tag{4.11}$$

where the linearly independent set of functions $\{f^{(\alpha)}\}$ satisfies the quasi-periodicity condition $f^{(\alpha)}(a/2)=f^{(\alpha)}(-a/2)e^{iqa}$. Then equations (3.10) and (3.13) reduce to

$$G^{(p)}=\sum_{n=1}^{\infty}\lambda_n^2|A_n^{(p)}|^2=\langle R\bar{\sigma}^{(p)\prime},\bar{\sigma}^{(p)\prime}\rangle, \tag{4.12}$$

$$\lambda_N^{(p)}=\sum_{n=1}^{\infty}\lambda_n|A_n^{(p)}|^2=\langle\bar{\sigma}^{(p)},\bar{u}^{(p)\prime}\rangle=\langle D\bar{\sigma}^{(p)},\bar{\sigma}^{(p)}\rangle. \tag{4.13}$$

To calculate $K^{(p)}$, define

$$v^{(p)}=\Lambda+\int_{-a/2}^{x}D(\xi)\bar{\sigma}^{(p)}(\xi)\,d\xi,$$
$$\Lambda=\int_{-a/2}^{a/2}D(\xi)\bar{\sigma}^{(p)}(\xi)\,d\xi/(e^{iqa}-1), \tag{4.14}$$

and from Parseval's equality relating to the expansion of $v^{(p)}$ in terms of $\{\varphi^{(n)}\}$ obtain

$$K^{(p)}=\sum_{n=1}^{\infty}|A_n^{(p)}|^2=\langle\rho v^{(p)},v^{(p)}\rangle. \tag{4.15}$$

Numerical results show that (4.4) gives very accurate bounds in this case; see Section 6.

For the one-dimensional case, the upper bound, however, can be improved considerably. In fact $\bar{\lambda}_N^{(1)}$ itself is an upper bound for the first eigenvalue. To see this, use the Rayleigh quotient (2.8) with $v^{(1)}$ as the trial function, arriving at

$$\lambda_1\leq\bar{\lambda}_R(v^{(1)})=\frac{\langle D\bar{\sigma}^{(1)},\bar{\sigma}^{(1)}\rangle}{\langle\rho v^{(1)},v^{(1)}\rangle}=\frac{\lambda_N^{(1)}}{K^{(1)}}=\bar{\lambda}_N^{(1)}. \tag{4.16}$$

Note that, since

$$\frac{\lambda_N^{(1)}}{K^{(1)}}=\frac{\sum_{n=1}^{\infty}\lambda_n|A_n^{(1)}|^2}{\sum_{n=1}^{\infty}|A_n^{(1)}|^2}\leq\frac{\sum_{n=1}^{\infty}\lambda_n^2|A_n^{(1)}|^2}{\sum_{n=1}^{\infty}\lambda_n|A_n^{(1)}|^2}=\frac{\langle R\bar{\sigma}^{(1)\prime},\bar{\sigma}^{(1)\prime}\rangle}{\langle D\bar{\sigma}^{(1)},\bar{\sigma}^{(1)}\rangle}=\bar{\bar{\lambda}}_R^{(1)}, \tag{4.17}$$

the upper bound (4.16) is superior to that given by (4.8).

Lower bound when ρ is continuous. In many cases it may happen that the mass-densities of various constituents of a composite are the same, while the corresponding elasticity coefficients differ considerably. In such a case, it may be useful to consider another lower bound which can be used for one-, two- and three-dimensional problems, provided that the exact eigenvalues are not too close to each other.

To obtain this bound, multiply (4.1) by $|A_n^{(p)}|^2\,\lambda_n$, sum on n, and arrive at

$$\sum_{n=1}^{\infty} \lambda_n^3|A_n^{(p)}|^2 - (\lambda_p + \lambda_{p+1})G^{(p)} + \lambda_p\lambda_{p+1}\lambda_N^{(p)} \geq 0. \tag{4.18}$$

Set

$$H^{(p)} = \sum_{n=1}^{\infty} \lambda_n^3|A_n^{(p)}|^2, \qquad \bar{H}^{(p)} = H^{(p)}/\lambda_N^{(p)}, \qquad \bar{\bar{G}}^{(p)} = G^{(p)}/\lambda_N^{(p)}. \tag{4.19}$$

Remark 6. If $H^{(p)}$ is finite and $\bar{\bar{G}}^{(p)} < \lambda_{p+1}$, then

$$\bar{\bar{G}}^{(p)} - \frac{\bar{H}^{(p)} - (\bar{\bar{G}}^{(p)})^2}{\lambda_{p+1} - \bar{\bar{G}}^{(p)}} \leq \lambda_p. \tag{4.20}$$

To use this bound, one must calculate $H^{(p)}$. When $R(\mathbf{x}) = 1/\rho(\mathbf{x})$ is continuous and, say, continuously differentiable in $\mathcal{R}$, and when the sequence of the approximating functions, $\{f^{(\alpha\beta\gamma)}\}$, is such that the approximate solution $\bar{\sigma}_{jk}^{(p)}$ is, say, twice continuously differentiable,† one can calculate $H^{(p)}$.

To this end observe from (3.3) and the completeness of $\{\psi_{jl}^{(n)}\}$ that the set of symmetric, tensor-valued functions, $\{\varphi_{(j,l)}^{(n)}\}$, is also complete, and that $\{\varphi_{(j,l)}^{(n)}/\sqrt{\lambda_n}\}$ is orthonormal in the following sense:

$$\langle C_{jlrs}\varphi_{(j,l)}^{(n)}, \varphi_{(r,s)}^{(m)}\rangle = \sqrt{(\lambda_n\lambda_m)}\,\delta_{nm}. \tag{4.21}$$

Now, define

$$[R\bar{\sigma}_{(jk,k}^{(p)}]_{,l)} = \tfrac{1}{2}\{[R\bar{\sigma}_{jk,k}^{(p)}]_{,l} + [R\bar{\sigma}_{lk,k}^{(p)}]_{,j}\}, \tag{4.22}$$

and obtain

$$-\langle C_{jlrs}[R\bar{\sigma}_{(jk,k}^{(p)}]_{,l)}, \varphi_{(r,s)}^{(n)}\rangle = \lambda_n^2 A_n^{(p)}, \text{(no sum on } n), \tag{4.23}$$

where $A_n^{(p)}$ is defined by (3.9). Hence one arrives at

$$-[R\bar{\sigma}_{(jk,k}^{(p)}]_{,l)} = \sum_{n=1}^{\infty} \lambda_n A_n^{(p)}\varphi_{(j,l)}^{(n)} \tag{4.24}$$

and the Parseval equality

$$\langle C_{jlmq}[R\bar{\sigma}_{(jk,k}^{(p)}]_{,l)}, [R\bar{\sigma}_{(mr,r}^{(p)}]_{,q)}\rangle = \sum_{n=1}^{\infty} \lambda_n^3|A_n^{(p)}|^2 = H^{(p)}. \tag{4.25}$$

†This is not a necessary condition. It suffices that $\langle C_{jlnq}(R\bar{\sigma}_{jk,k}^{(p)})_{,l}, (R\bar{\sigma}_{nr,r}^{(p)})_{,q}\rangle$ be finite.

Note that the condition $\bar{\bar{G}}^{(p)} < \lambda_{p+1}$ places a severe restriction on the usefulness of the lower bound (4.20); for illustrative examples see Section 6.

New quotient as upper bound. Consider the special case when the mass-density ρ is constant; for simplicity set $\rho = 1$. Choose the approximating functions (2.21), and observe from equation (2.12) that

$$\bar{\sigma}^{(p)}_{jk,k} = \sum_{\alpha,\beta,\gamma=1}^{M} W_j^{(\alpha\beta\gamma)(p)} f^{(\alpha\beta\gamma)}, \tag{4.26}$$

where $W_j^{(\alpha\beta\gamma)(p)}$ are constants. From (4.26) and (2.13), it now follows that

$$\begin{aligned}\langle \bar{\sigma}^{(p)}_{jk,k}, \bar{\sigma}^{(q)}_{jl,l}\rangle &= \lambda_N^{(p)}\langle \bar{u}^{(p)}_{j,l}, \bar{\sigma}^{(q)}_{jl}\rangle \\ &= \lambda_N^{(p)}\langle D_{jkmn}\bar{\sigma}^{(p)}_{mn}, \bar{\sigma}^{(q)}_{jk}\rangle \\ &= \lambda_N^{(p)}\delta_{pq},\end{aligned} \tag{4.27}$$

where $(4{\cdot}27)_1$ is obtained from $(2.13)_1$, $(4.27)_2$ from $(2.13)_2$, and $(4.27)_3$ from $(2.15)_2$. Now, from the stress Rayleigh quotient (4.7) with $R = \rho = 1$, one deduces that

$$\bar{\bar{\lambda}}_R^{(p)} = \lambda_N^{(p)} = \frac{\langle \bar{\sigma}^{(p)}_{jk,k}, \bar{\sigma}^{(p)}_{jl,l}\rangle}{\langle D_{jkmn}\bar{\sigma}^{(p)}_{mn}, \bar{\sigma}^{(p)}_{jk}\rangle}. \tag{4.28}$$

Remark 7. With test functions (2.21) and when ρ is constant, the minimization of the new quotient gives upper bounds, i.e.

$$\lambda_N^{(p)} \geq \lambda_p. \tag{4.29}$$

Proof. Consider the Rayleigh quotient (4.7) and a solution in the form

$$\sigma_{ij} = \sum_{p=1}^{\bar{M}} \xi_p \bar{\sigma}^{(p)}_{jk}. \tag{4.30}$$

From $(2.15)_2$ it follows that $\bar{\sigma}^{(p)}_{jk}$, $p = 1, 2, \ldots, \bar{M}$, are linearly independent functions which, according to (2.21), satisfy the boundary conditions $(2.3)_2$. Hence minimization of (4.7) with (4.30) as the test function and ξ_p's as the unknown parameters yields, in view of Poincare's minimum-maximum principle,† upper bounds for each of the first $\bar{M}$ eigenvalues. But because of (4.27) and (4.28), the corresponding approximate eigenvalues are given by $\lambda_N^{(p)}$. Hence one arrives at (4.29).

Remark 8. With test functions (2.21) and when the elasticity tensor C_{ijkl} is constant, the minimization according to (2.13) of the new quotient, yields upper bounds equal to those obtained by the minimization oı the displacement Rayleigh quotient (2.8).

The proof of this assertion follows immediately from equation $(2.13)_2$ which shows that, for constant compliance tensor D_{klmn}, one has

$$\bar{u}^{(p)}_{(k,l)} = D_{klmn}\bar{\sigma}^{(p)}_{mn} \tag{4.31}$$

†The same result follows from Courant's maximum-minimum principle[15].

for each $p = 1, 2, \ldots, 2M+1$. Remark 3, expression (2.20), then implies that $\bar{\lambda}_R^{(p)} = \lambda_N^{(p)}$, and since a Rayleigh–Ritz procedure is involved, one arrives at upper bounds.

5. ESTIMATE OF ERRORS

It is known that if the square root of the mean-square error in satisfying (2.5) is ϵ, then the error in the Rayleigh quotient will not exceed ϵ^2. At this point, the same result cannot be presented for the new quotient (2.11). But, one can obtain a certain estimate for the corresponding errors.

To this end, define the error functions

$$D_{jklm}\bar{\sigma}_{jk}^{(p)} - \bar{u}_{(l,m)}^{(p)} = e_{(lm)}^{(1)} = \tfrac{1}{2}[e_{lm}^{(1)} + e_{ml}^{(1)}],$$
$$\bar{\sigma}_{jk,k}^{(p)} + \lambda_N^{(p)}\rho\bar{u}_j^{(p)} = e_j^{(2)}, \tag{5.1}$$

and calculate the average errors

$$\langle C_{jklm}e_{(lm)}^{(1)}, e_{(jk)}^{(1)}\rangle^{1/2} = \epsilon_1, \qquad \langle Re_j^{(2)}, e_j^{(2)}\rangle^{1/2} = \epsilon_2. \tag{5.2}$$

From (3.1), (3.8) and (3.9), it follows that

$$Re_j^{(2)} = R\bar{\sigma}_{jk,k}^{(p)} + \lambda_n^{(p)}\bar{u}_j^{(p)} = \sum_{n=1}^{\infty} (\lambda_N C_n^{(p)} - \lambda_n A_n^{(p)})\varphi_j^{(n)}, \tag{5.3}$$

and hence

$$\epsilon_2^2 = \sum_{n=1}^{\infty} |\lambda_N^{(p)} C_n^{(p)} - \lambda_n A_n^{(p)}|^2. \tag{5.4}$$

Moreover, one has

$$C_{jklm}e_{(lm)}^{(1)} = \bar{\sigma}_{jk}^{(p)} - C_{jkmn}\bar{u}_{(m,n)}^{(p)} = \sum_{n=1}^{\infty} (A_n^{(p)} - C_n^{(p)})\psi_{jk}^{(n)}, \tag{5.5}$$

and hence

$$\epsilon_1^2 = \sum_{n=1}^{\infty} \lambda_n |A_n^{(p)} - C_n^{(p)}|^2. \tag{5.6}$$

Equations (5.4) and (5.6), together with equations (2.13), suggest reasons for the effectiveness of the new quotient.

To begin with, one observes that *if* ϵ_1 and ϵ_2 approach zero as M becomes large (i.e. as more terms are included in the approximate solutions (2.10) and (2.12)), then from (5.4) and (5.6) it follows that

$$|A_n^{(p)} - C_n^{(p)}| \to 0 \text{ and } |\lambda_N^{(p)} C_n^{(p)} - \lambda_n A_n^{(p)}| \to 0 \text{ as } \epsilon_1, \epsilon_2 \to 0.$$

Hence if $\lambda_N^{(n)} \neq \lambda_n$, then $A_n^{(p)} \to C_n^{(p)} \to 0$. On the other hand, since $\sum_{n=1}^{\infty} |C_n^{(p)}|^2 = 1$, not all $C_n^{(p)}$'s can vanish. For $C_p^{(p)} \neq 0$, then one obtains

$$A_p^{(p)} \to C_p^{(p)} \to 1 \text{ and } \lambda_N^{(p)} \to \lambda_p \text{ as } \epsilon_1, \epsilon_2 \to 0.$$

Now assume that the approximating functions $f^{(\alpha\beta\gamma)}$ are such that† the error functions $e_{(jk)}^{(1)}$ and $e_j^{(2)}$ admit the following Fourier series:

$$e_{(jk)}^{(1)} = \sum_{\alpha,\beta,\gamma=1}^{\infty} E_{(jk)}^{(\alpha\beta\gamma)} f^{(\alpha\beta\gamma)}, \qquad E_{(jk)}^{(\alpha\beta\gamma)} = \langle e_{(jk)}^{(1)}, f^{(\alpha\beta\gamma)} \rangle,$$

$$e_j^{(2)} = \sum_{\alpha,\beta,\gamma=1}^{\infty} E_j^{(\alpha\beta\gamma)} f^{(\alpha\beta\gamma)}, \qquad E_j^{(\alpha\beta\gamma)} = \langle e_j^{(2)}, f^{(\alpha\beta\gamma)} \rangle, \tag{5.7}$$

and that the corresponding coefficients satisfy Parseval's equalities

$$\langle e_{(jk)}^{(1)}, e_{(jk)}^{(1)} \rangle = \sum_{\alpha,\beta,\gamma=1}^{\infty} |E_{(jk)}^{(\alpha\beta\gamma)}|^2 \text{ and } \langle e_j^{(2)}, e_j^{(2)} \rangle = \sum_{\alpha,\beta,\gamma=1}^{\infty} |E_j^{(\alpha\beta\gamma)}|^2, \tag{5.8}$$

respectively. Equations (2.13) show that the first $6M^3$ terms in the right-hand side of $(5.8)_1$, and the first $3M^3$ terms in the right-hand side of $(5.8)_2$, are identically zero, i.e. the corresponding errors have zero projections on the first M^3 coordinate functions $f^{(\alpha\beta\gamma)}$. This suggests that, as M becomes unbounded, the mean-square errors (5.8), and consequently the errors ϵ_1 and ϵ_2 in (5.2) should approach zero, provided that the approximating set $\{f^{(\alpha\beta\gamma)}\}$ is suitably chosen. Note that since $e_{(lm)}^{(1)}$ and $e_j^{(2)}$ are dependent on M, no definite conclusion can be deduced from (2.13) and (5.8). These equations are not, by themselves, sufficient to guarantee that ϵ_1 and ϵ_2 vanish with increasing M.

Remark 9. When $\rho = \text{constant}$, then $\lambda_N^{(p)} \to \lambda_p$ as $M \to \infty$.

The proof of this assertion follows from the fact that the approximate solution $\bar{\sigma}_{ij}^{(p)}$ minimizes the Rayleigh quotient (4.7) in a proper Rayleigh–Ritz procedure (see Remark 7).

Whether ρ is constant or not, one can obtain an error estimate for $\lambda_N^{(p)}$ using (5.4) as follows:

$$|\lambda_N^{(p)} C_p^{(p)} - \lambda_p A_p^{(p)}| \geq |\lambda_N^{(p)} - \lambda_p||C_p^{(p)}| - \lambda_p |A_p^{(p)} - C_p^{(p)}|.$$

Then from (5.6) one obtains

$$|C_p^{(p)}||\lambda_N^{(p)} - \lambda_p| \leq \sqrt{(\lambda_p)}\epsilon_1 + \epsilon_2. \tag{5.9}$$

Since ϵ_1 and ϵ_2 can be calculated from (5.2), (5.9) provides an estimate for the accuracy of $\lambda_N^{(p)}$; instead of λ_p in the right-hand side of (5.9), one may use $\lambda_N^{(p)}$.

6. NUMERICAL RESULTS

The theory presented in the preceding sections will now be applied to composites whose unit cells consist of two constituents, each homogeneous and isotropic. Waves propagating normal to layers in a layered composite will be considered first, since the corresponding elasticity problem can be solved exactly, and hence provides a good check on the accuracy of the results. Then waves propagating normal to fibers in fiber-reinforced composites with rectangular (Fig. 1), and elliptical (Fig. 2), fibers will be treated.

†For example, normalized expressions (2.21) are used.

For all the problems mentioned above, equation (2.13) with test functions (2.21) can be written as

$$\mathbf{\Omega U} + \mathbf{HS} = \mathbf{0}, \qquad \mathbf{H^*U} + \mathbf{\Phi S} = \mathbf{0}, \tag{6.1}$$

which can be solved to yield

$$\mathbf{S} = -\mathbf{\Phi}^{-1}\mathbf{H^*U}, \qquad [\mathbf{\Omega} - \mathbf{H\Phi}^{-1}\mathbf{H^*}]\mathbf{U} = \mathbf{0}, \tag{6.2}$$

where **U** and **S** are vectors of the unknown coefficients in the approximate expressions for the displacement and the stress fields, respectively. The other quantities in equations (6.1) and (6.2) depend on the specific case, as discussed below.

Layered composites. Let the unit cell of length a be composed of two materials, one occupying region $-a/2 \geq x \geq -b/2$ and $b/2 \leq x \leq a/2$ and having mass-density ρ_1 and elastic constant η_1, the other occupying the region $|x| \leq b/2$ and having mass-density ρ_2 and elasticity constant η_2; $D_1 = 1/\eta_1$ and $D_2 = 1/\eta_2$. For functions $f^{(\alpha)}$ choose $e^{i(Q+2\pi\alpha)\xi}$, where $\xi = x/a$, and $Q = qa$. The unknowns **U** and **S** are $(2M'+1)$-dimensional vectors with components $U^{(\alpha)}$ and $S^{(\alpha)}$, $\alpha = 0, \pm 1, \ldots, \pm M'$; in (4.11), $M = 2M'+1$. The matrix $\mathbf{\Omega} = [\Omega_{mn}]$ and the diagonal matrix $\mathbf{H} = [H_{mn}]$ are defined by

$$\mathbf{\Omega} = \nu^2\left(n_1 + \frac{n_2}{\gamma}\right)(n_1 + \gamma n_2)\bar{\mathbf{\Omega}}$$

$$\bar{\Omega}_{mn} = \begin{cases} \dfrac{\theta - 1}{n_1 + \theta n_2}\dfrac{\sin[\pi(m-n)n_2]}{\pi(m-n)} \text{ if } m \neq n, \\ 1 \text{ if } m = n, \end{cases}$$

$$H_{mm} = i(Q + 2\pi m), \tag{6.3}$$

and the matrix $\mathbf{\Phi}$ is obtained by replacing in matrix $\mathbf{\Omega}$, $(\theta-1)/(n_1+\theta n_2)$ by $(1-\gamma)/(\gamma n_1 + n_2)$. In equation (6.3), the following notation is used:

$$\nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2}, \quad \bar{\eta} = n_1\eta_1 + n_2\eta_2, \quad \bar{\rho} = n_1\rho_1 + n_2\rho_2,$$

$$n_1 = \frac{a-b}{a}, \quad n_2 = \frac{b}{a}, \quad \gamma = \eta_2/\eta_1, \quad \theta = \rho_2/\rho_1, \tag{6.4}$$

where $\omega(=\sqrt{\lambda})$ is the wave frequency, and ν is the dimensionless frequency which is to be calculated as a function of the dimensionless wave number Q.

With all quantities in (6.2) fixed, one obtains the approximate values of the first $2M'+1$ eigenfrequencies by setting the determinant of the coefficients of **U** in $(6.2)_2$ equal to zero. Equations (6.2) then give $\mathbf{U}^{(p)}$ and $\mathbf{S}^{(p)}$ for the pth mode. Note that for the actual calculation it is not necessary to normalize $\bar{u}$.

For the calculation of the bounds, one first obtains†

†Note that in order to obtain $G^{(p)}$, one uses in (6.5) $\mathbf{U}^{(p)}$ and $\mathbf{S}^{(p)}$ which are **U** and **S** with components $U^{(\alpha)}$ and $S^{(\alpha)}$ corresponding to the pth mode.

$$G = \bar{R}\frac{\sum_{\substack{\alpha,\beta=0\\ \alpha\neq\beta}}^{\pm M'} S^{(\alpha)}S^{(\beta)*}\frac{1-\theta}{\theta n_1+n_2}(Q+2\pi\alpha)(Q+2\pi\beta)\frac{\sin[\pi(\alpha-\beta)n_2]}{\pi(a-\beta)}+\sum_{\alpha=0}^{\pm M'}|S^{(\alpha)}|^2(Q+2\pi\alpha)^2}{\sum_{\substack{\alpha,\beta=0\\ \alpha\neq\beta}}^{\pm M'} U^{(\alpha)}U^{(\beta)*}\frac{\theta-1}{n_1+\theta n_2}\frac{\sin[\pi(\alpha-\beta)n_2]}{\pi(\alpha-\beta)}+\sum_{\alpha=0}^{\pm M'}|U^{(\alpha)}|^2}, \tag{6.5}$$

where

$$\bar{R} = (\bar{\eta}/\bar{\rho}a^2)^2(n_1+n_2/\theta)(n_1+\theta n_2)[(n_1+n_2/\gamma)(n_1+\gamma n_2)]^{-2}. \tag{6.6}$$

Then, one calculates $K = \langle \rho v, v\rangle$, arriving at

$$\begin{aligned} K &= \sum_{n=1}^{\infty}|A_n|^2\\ &= a^3\rho_1 D_1{}^2\Bigg[\sum_{\substack{\alpha,\beta=0\\ \alpha\neq\beta}}^{\pm M'} S^{(\alpha)}S^{(\beta)*}\left\{K_1+\frac{(\theta/\gamma^2-1)}{(Q+2\pi\alpha)(Q+2\pi\beta)}\frac{\sin[\pi(\alpha-\beta)n_2]}{\pi(\alpha-\beta)}\right\}\\ &\quad+\sum_{\alpha=0}^{\pm M'}|S^{(\alpha)}|^2\left\{K_1+\frac{n_1+n_2\theta/\gamma^2}{(Q+2\pi\alpha)^2}\right\}\Bigg], \end{aligned} \tag{6.7}$$

where

$$\begin{aligned} K_1 = \Bigg\{&\left[\left(\frac{1}{\gamma}-1\right)\sin[(Q+2\pi\alpha)n_2/2]+\sin[(Q+2\pi\alpha)/2]\right]\frac{(n_1+n_2\theta)}{\sin(Q/2)}\\ &+4\left[\left(\frac{\theta}{\gamma}-1\right)\sin[(Q+2\pi\alpha)n_2/2]+\sin[(Q+2\pi\alpha)/2]\right]\frac{\cos(Q/2)}{(Q+2\pi\alpha)}\\ &-2\Bigg[(n_1+n_2\theta)\cos(\pi\alpha)+\left(\frac{1}{\gamma}-1\right)(1-n_2)\sin(Q/2)\sin[(Q+2\pi\alpha)n_2/2]\\ &+\left(\frac{1}{\gamma}-1\right)n_2\theta\cos[Q(1-n_2)/2-\pi\alpha n_2]\Bigg]\Bigg\}\Bigg[\left(\frac{1}{\gamma}-1\right)\sin[(Q+2\pi\beta)n_2/2]\\ &+\sin[(Q+2\pi\beta)/2]\Bigg]/[(Q+2\pi\alpha)(Q+2\pi\beta)\sin(Q/2)]\\ &+4\Bigg\{[\cos[(Q+2\pi\alpha)n_2/2]-\cos[(Q+2\pi\alpha)/2]]\left(\frac{1}{\gamma}-1\right)\sin[(Q+2\pi\beta)n_2/2]\\ &+[\sin[(Q+2\pi\alpha)n_2/2]-\sin[(Q+2\pi\alpha)/2]]\cos[(Q+2\pi\beta)/2]\\ &-\frac{\theta}{\gamma}\sin[(Q+2\pi\alpha)n_2/2]\Bigg[\left(\frac{1}{\gamma}-1\right)\cos[(Q+2\pi\beta)n_2/2]\\ &+\cos[(Q+2\pi\beta)/2]\Bigg]\Bigg\}/[(Q+2\pi\alpha)^2(Q+2\pi\beta)]\\ &+\Bigg\{2(1-n_2)\left(\frac{1}{\gamma}-1\right)\sin[(Q+2\pi\alpha)n_2/2][\sin[(Q+2\pi\beta)/2]\\ &+\left(\frac{1}{\gamma}-1\right)\sin[(Q+2\pi\beta)n_2/2]\Bigg]+(n_1+n_2\theta)\cos[\pi(\alpha-\beta)]\\ &+n_2\theta\left(\frac{1}{\gamma}-1\right)\Bigg[\left(\frac{1}{\gamma}-1\right)\cos[\pi(\alpha-\beta)n_2]\\ &+2\cos[Q(1-n_2)/2-\pi(\beta-\alpha n_2)]\Bigg]\Bigg\}/[(Q+2\pi\alpha)(Q+2\pi\beta)]. \end{aligned} \tag{6.8}$$

Direct substitution into (4.4) now gives lower and upper bounds for ν^2 which relate to ω^2 by equation $(6.4)_1$, where for λ_{p+1} and λ_{p-1} the corresponding approximate values may be used.

Numerical results are given in Table 1 for $\eta_2/\eta_1 = 100$, $\rho_2/\rho_1 = 3$, and indicated values of Q and M'. In this table, $\nu^{(l)}$, $\nu^{(u)}$, $\bar{\nu}_R$, and ν_N, respectively, refer to the lower bound, upper bound calculated from (4.4), the values of ν obtained from the Rayleigh quotient (2.8), and the value given by the new quotient (2.11). The quantity $\bar{\nu}_N = (\nu_N/K)$ which is the dimensionless form of $\bar{\lambda}_N$, see (4.16), is also reported in this table. As is seen $\bar{\nu}_N$ is a very close approximation for the exact ν. These results strongly suggest that $\bar{\nu}_N$ should be upper bound for all corresponding eigenvalues, *but this is true only for the first eigenvalue.*

Table 1 is for $M' = 1$, i.e. the crudest approximation, and for $M' = 5$. Here, for each value of Q, the corresponding values of the wave frequency for the first two and for the first five modes are listed, for $M' = 1$ and $M' = 5$, respectively. As is seen even for $M' = 1$ the bounds are very good. The bounds, as well as the value of ν_N improve as M' is increased. This is illustrated for $M' = 5$.

Table 1. Layered composites. Eigenfrequency ν and its lower and upper bounds for first two modes: $n_1 = n_2 = \frac{1}{2}$, $\theta = 3$ and $\gamma = 100$

Q	Lower bound ν_l	New quotient ν_N	Equation (4.16)† $\bar{\nu}_N = \frac{\nu_N}{K}$	Exact ν	Upper bound (4.4) ν_u	Rayleigh quotient $\bar{\nu}_R$
			$M' = 1$			
1·0	0·1932	0·1934	0·1933	0·1933	0·1999	0·4604
	1·3509	1·3548	1·3629	1·3616	1·3951	5·7947
2·0	0·3534	0·3541	0·3546	0·3544	0·3688	0·8541
	1·2796	1·2776	1·2886	1·2873	1·3204	5·1731
3·0	0·4317	0·4297	0·4341	0·4336	0·4510	1·0831
	1·2251	1·2462	1·2403	1·2386	1·3066	4·7414
			$M' = 5$			
	0·193	0·193	0·193	0·193	0·196	0·210
	1·361	1·361	1·362	1·362	1·364	1·633
1·0	2·495	2·497	2·497	2·497	2·501	3·014
	3·754	3·771	3·773	3·773	3·797	5·079
	4·909	4·938	4·945	4·942	4·995	6·683
	0·354	0·354	0·354	0·354	0·359	0·384
	1·287	1·287	1·287	1·287	1·289	1·556
2·0	2·543	2·545	2·545	2·545	2·549	3·072
	3·713	3·728	3·730	3·730	3·756	4·991
	4·958	4·985	4·996	4·993	5·041	6·847
	0·433	0·434	0·434	0·434	0·440	0·467
	1·238	1·239	1·239	1·239	1·239	1·505
3·0	2·572	2·572	2·574	2·573	2·576	3·116
	3·687	3·704	3·705	3·703	3·733	4·931
	4·992	5·002	5·029	5·025	5·069	7·020

†For each value of Q, only the first eigenvalue calculated from (4.16) will in general be an upper bound for the corresponding exact eigenvalue.

†The test functions (2.21) now become

$$f^{(\alpha\beta)} = e^{i(q_1x_1+q_2x_2+2\pi\alpha x_1+2\pi\beta x_2)}.$$

In this section it is assumed that $q_2 = 0$ and $q_1 = q$.

Fiber-reinforced composites. The quantities which enter equations (6.1) and (6.2) will now have the following definitions.† $\mathbf{U} = \{\mathbf{U}_1, \mathbf{U}_2\}^T$ is a $2(2M'+1)^2$-dimensional vector with components $U_j^{(\alpha\beta)}$, $j = 1, 2$, $\alpha, \beta = 0, \pm 1, \ldots, \pm M'$, where superposed T denotes transpose; $\mathbf{S} = \{\mathbf{S}_{11}, \mathbf{S}_{12}, \mathbf{S}_{22}\}^T$ is a $3(2M'+1)^2$-dimensional vector with components $S_{jk}^{(\alpha\beta)}$; the matrix $\mathbf{\Omega} = [\Omega(I, J)]$ is $2(2M'+1)^2$ by $2(2M'+1)^2$; $\mathbf{\Phi} = [\Phi(I, J)]$ is $3(2M'+1)^2$ by $3(2M'+1)^2$; and $\mathbf{H} = -\bar{\mathbf{H}}^T = -[\bar{H}(I, J)]^T$ is $2(2M'+1)^2$ by $3(2M'+1)^2$.

The matrix $\mathbf{\Omega}$ is defined as follows: (i) For $I_1 = (\alpha + 1 + M') + (\beta + M')(2M'+1)$ and $J_1(\gamma + 1 + M') + (\delta + M')(2M'+1)$, $\alpha, \beta, \gamma, \delta = 0, \pm 1, \ldots \pm M'$, one has:
(a) composites with rectangular fibers (Fig. 1),

$$\Omega(I_1, J_1) = \frac{\nu^2}{d}\begin{cases} \dfrac{(\theta-1)}{(\bar{n}_1 + \bar{n}_2\theta)} \dfrac{\sin \pi(\delta-\beta)m_2}{\pi(\delta-\beta)} \dfrac{\sin \pi(\gamma-\alpha)n_2}{\pi(\gamma-\alpha)}, & \alpha \neq \gamma,\ \beta \neq \delta, \\[2ex] \dfrac{(\theta-1)n_2 \sin \pi(\delta-\beta)m_2}{(\bar{n}_1 + \bar{n}_2\theta)\pi(\delta-\beta)}, & \alpha = \gamma,\ \beta \neq \delta, \\[2ex] \dfrac{(\theta-1)m_2 \sin \pi(\alpha-\gamma)n_2}{(\bar{n}_1 + \bar{n}_2\theta)\pi(\alpha-\gamma)}, & \alpha \neq \gamma, \beta = \delta, \\[2ex] 1, & \alpha = \gamma,\ \beta = \delta; \end{cases} \tag{6.9}$$

(b) composites with elliptical fibers (Fig. 2),

$$\Omega(I_1, J_1) = \frac{\nu^2}{d}\begin{cases} \dfrac{(\theta-1)}{(\bar{n}_1 + \bar{n}_2\theta)} \dfrac{m_2 J_1(R)}{2\left\{(\alpha-\gamma)^2 + \left[(\beta-\delta)\dfrac{m_2}{n_2}\right]^2\right\}^{1/2}} & \text{if } \alpha \neq \gamma, \text{ and/or } \beta \neq \delta, \\[2ex] 1 & \text{if } \alpha = \gamma,\ \beta = \delta, \end{cases} \tag{6.10}$$

where† $J_1(R)$ is the Bessel function of the first kind and first order with argument

$$R = \pi\{n_2^2(\alpha-\gamma)^2 + m_2^2(\beta-\delta)^2\}^{1/2}.$$

(ii) For $I_2 = I_1 + (2M'+1)^2$ and $J_2 = J_1 + (2M'+1)^2$, one has

$$\Omega(I_2, J_2) = \Omega(I_1, J_1), \quad \text{and} \quad \Omega(I_2, J_1) = \Omega(I_1, J_2) = 0. \tag{6.11}$$

In these equations, the following notation is used:‡

$$\nu^2 = \omega^2 a_1^2 \bar{\rho}/\bar{C}_{1111}, \qquad \bar{\rho} = \rho^{(1)}\bar{n}_1 + \rho^{(2)}\bar{n}_2, \qquad \bar{C}_{1111} = C_{1111}^{(1)}\bar{n}_1 + C_{1111}^{(2)}\bar{n}_2,$$
$$\bar{n}_1 = 1 - \bar{n}_2, \qquad \bar{n}_2 = \frac{b_1 b_2}{a_1 a_2}, \qquad \theta = \frac{\rho^{(2)}}{\rho^{(1)}}, \qquad n_2 = \frac{b_1}{a_1}, \qquad m_2 = \frac{b_2}{a_2},$$
$$d = 1/(\bar{C}_{1111}\bar{D}_{1111}), \qquad \bar{D}_{1111} = D_{1111}^{(1)}\bar{n}_1 + D_{1111}^{(2)}\bar{n}_2. \tag{6.12}$$

The matrix $\mathbf{\Phi}$ is defined as follows: (i) For $I_1 = (\alpha + 1 + M') + (\beta + M')(2M'+1)$ and $J_1 = (\gamma + 1 + M') + (\delta + M')(2M'+1)$, $\Phi(I_1, J_1)$ is obtained from (6.9) if one replaces $(\theta - 1)/(\bar{n}_1 +$

†Note that J_1 is an integer, while $J_1(R)$ is the Bessel function.

‡For elliptical fibers $\bar{n}_2$ must be replaced by $\dfrac{\pi}{4}\dfrac{b_1 b_2}{a_1 a_2}$.

$\bar{n}_2\theta$) by $(\gamma_{1111}-1)/(\bar{n}_1+\bar{n}_2\gamma_{1111})$ and omits ν^2/d; (ii) For $I_2=I_1+(2M'+1)^2$ and $J_2=J_1+(2M'+1)^2$, and when $\alpha\neq\gamma$ or $\beta\neq\delta$, $\Phi(I_2, J_2)$ is obtained from (6.9) if one replaces $(\nu^2)/(d)$ $(\theta-1)/(\bar{n}_1+\bar{n}_2\theta)$ by $4[(\gamma_{1212}-1)R_{1212}/(\bar{n}_1+\bar{n}_2\gamma_{1111})]$. When $\alpha=\gamma$ and $\beta=\delta$, then for composites with rectangular fibers one has $\Phi(I_2, J_2)=4[(\bar{n}_1+\bar{n}_2\gamma_{1212})R_{1212}/(\bar{n}_1+\bar{n}_2\gamma_{1111})]$. Also $\Phi(I, J_2)=\Phi(I_2, J_1)=0$. (iii) For $I_3=I_2+(2M'+1)^2$ and $J_3=J_2+(2M'+1)^2$, one has $\Phi(I_3, J_3)=\Phi(I_1, J_1)$, $\Phi(I_2, J_3)=\Phi(I_3, J_2)=0$, $\Phi(I_1, J_3)$ is obtained from $\Phi(I_2, J_2)$ if one replaces in the latter $4R_{1212}$ and γ_{1212} by R_{1122} and γ_{1122}, respectively, and $\Phi(I_3, J_1)=\Phi(I_1, J_3)$. Here, the following notation is used:

$$\gamma_{jklm}=\frac{D^{(2)}_{jklm}}{D^{(1)}_{jklm}},\qquad R_{jklm}=\frac{D^{(1)}_{jklm}}{D^{(1)}_{1111}},\qquad j,k,l,m \text{ not summed.} \tag{6.13}$$

The matrix $\bar{\mathbf{H}}$ is defined as follows†

$$\bar{H}(I_1, J_1)=-i\begin{cases}Q+2\pi\gamma & \text{if } \alpha=\gamma \text{ and } \beta=\delta,\\ 0 & \text{if } \alpha\neq\gamma \text{ or } \beta\neq\delta,\end{cases}$$

$$\bar{H}(I_2, J_1)=-i\begin{cases}2\pi\delta n_o & \text{if } \alpha=\gamma \text{ and } \beta=\delta,\\ 0 & \text{if } \alpha\neq\gamma \text{ or } \beta\neq\delta,\end{cases}$$

$$\bar{H}(I_2, J_2)=\bar{H}(I_1, J_1),\qquad \bar{H}(I_3, J_2)=\bar{H}(I_2, J_1),\qquad \bar{H}(I_1, J_2)=\bar{H}(I_3, J_1)=0, \tag{6.14}$$

where

$$n_o=\frac{a_1}{a_2}.$$

To obtain an upper bound for the first eigenvalue, one calculates

$$\bar{\bar{\lambda}}_R^{(1)}=\langle R\sigma^{(1)}_{jk,k}, \sigma^{(1)}_{jl,l}\rangle/\langle D_{jkpq}\sigma^{(1)}_{jk}, \sigma^{(1)}_{pq}\rangle=\bar{R}B/T, \tag{6.15}$$

where

$$B=\sum_{\substack{\alpha,\beta,\gamma,\delta=0\\ \alpha\neq\gamma,\beta\neq\delta}}^{\pm M'} S\frac{(1-\theta)}{(\bar{n}_1\theta+\bar{n}_2)}B_1+\sum_{\substack{\alpha,\beta,\gamma,\delta=0\\ \alpha=\gamma,\beta\neq\delta}}^{\pm M'} S\frac{(1-\theta)n_2}{(\bar{n}_1\theta+\bar{n}_2)}B_2$$
$$+\sum_{\substack{\alpha,\beta,\gamma,\delta=0\\ \alpha\neq\gamma,\beta=\delta}}^{\pm M'} S\frac{(1-\theta)m_2}{(\bar{n}_1\theta+\bar{n}_2)}B_3+\sum_{\substack{\alpha,\beta,\gamma,\delta=0\\ \alpha=\gamma,\beta=\delta}}^{\pm M'} S, \tag{6.16}$$

$$S=S^{(\alpha\beta)}_{k1}S^{(\gamma\delta)*}_{k1}(Q+2\pi\alpha)(Q+2\pi\gamma)+S^{(\alpha\beta)}_{k2}S^{(\gamma\delta)*}_{k2}(4\beta\delta\pi^2n_o^2)$$
$$+S^{(\alpha\beta)}_{k2}S^{(\gamma\delta)*}_{k1}(2\pi\beta n_o(Q+2\pi\gamma))+S^{(\alpha\beta)}_{k1}S^{(\gamma\delta)*}_{k2}(2\pi\delta n_o(Q+2\pi\alpha)), \tag{6.17}$$

and where B_1, B_2 and B_3 are defined as follows.

†Note that the vector $\mathbf{q}$ has components $q_1=q$, $q_2=0$. Then $Q=qa_1$.

(a) composites with rectangular fibers (Fig. 1),

$$B_1 = \frac{\sin[\pi(\alpha-\gamma)n_2]}{\pi(\alpha-\gamma)}\frac{\sin[\pi(\beta-\delta)m_2]}{\pi(\beta-\delta)}, \qquad B_2 = \frac{\sin[\pi(\beta-\delta)m_2]}{\pi(\beta-\delta)},$$

$$B_3 = \frac{\sin[\pi(\alpha-\gamma)n_2]}{\pi(\alpha-\gamma)}; \tag{6.18}$$

(b) composites with elliptical fibers (Fig. 2),

$$B_1 = B_2 = B_3 = \frac{m_2 J_1(R)}{2\left\{(\alpha-\gamma)^2 + \left[(\beta-\delta)\frac{m_2}{n_2}\right]^2\right\}^{1/2}}. \tag{6.19}$$

In equation (6.15),

$$T = \mathbf{S}^T \mathbf{\Phi} \mathbf{S}. \tag{6.20}$$

The dimensionless upper bound given by (6.15) is defined by

$$\bar{\bar{\nu}}_R = \{\bar{\bar{\lambda}}_R a_1^2 \bar{\rho}/\bar{C}_{1111}\}^{1/2}. \tag{6.21}$$

Numerical results are presented in Table 2 for square fibers, i.e. $(b_2/b_1) = 1$, for rectangular fibers, i.e. $(b_2/b_1) = (3/2)$, for circular fibers, and for elliptical fibers, i.e. $(b_2/b_1) = 2$. All these results are for $\theta = 3$, $(C^{(2)}_{1111}/C^{(1)}_{1111}) = 100$, $(b_1/a_1) = \frac{1}{2}$, $n_o = (a_1/a_2) = 1$ and $M' = 1$. The Poisson Ratio of the two constituents is taken to be 0·3. This table lists, for the indicated values of the wave number Q, the first five eigenfrequencies calculated from the new quotient, ν_N, together with their upper bounds calculated from (4.7) and (6.21). For comparison, the results obtained from the Rayleigh quotient (2.8) are also listed. Note that, for each value of Q, only the *first* approximate $\bar{\bar{\nu}}_R$ was shown in Section 4 to be an upper bound for the corresponding exact eigenvalue. This situation and the bounds can be improved as follows.

Upper bounds for second and higher eigenvalues. Consider the Rayleigh quotient (4.7) and choose for the test function equation (4.30) in which $\bar{\sigma}^{(p)}_{jk}$ are given by

$$\bar{\sigma}^{(p)}_{jk} = \sum_{\alpha,\beta=0}^{\pm M'} S^{(\alpha\beta)(p)}_{jk} e^{i(qx_1 + 2\pi\alpha x_1 + 2\pi\beta x_2)}, \tag{6.22}$$

where $S^{(\alpha\beta)(p)}_{jk}$ are components of the vector $\mathbf{S}^{(p)}$ obtained from equation $(6.2)_1$. Minimization of $\bar{\bar{\lambda}}_R$ with respect to the unknown coefficients ξ_p now yields upper bounds for $(2M'+1) = M$ first eigenvalues. In Table 2 these now bounds are denoted by $\bar{\bar{\nu}}^*_R$.

Lower and upper bounds when ρ = const. In this case the new quotient yields upper bounds for each of the first $M = 2M'+1$ eigenvalues, as was shown in Remark 7; see equation (4.29). To obtain lower bounds, one may use the result of Remark 6, inequality (4.20). To this end, one calculates the quantity H, equations (4.19) and (4.25), arriving at

$$H = \frac{d^2}{(a_1^2\rho/\bar{C}_{1111})^3}\tilde{H}, \qquad \bar{H} = \frac{H}{\lambda_N} = \frac{1}{\nu_N^2}\frac{d^2}{(a_1^2\rho/\bar{C}_{1111})^2}\tilde{H},$$

Table 2. Fiber-reinforced composites. Eigenfrequency ν and its upper bounds for first five modes: $M'=1$, $\theta=3$, $(C^{(2)}_{1111}/C^{(1)}_{1111})=100$, both Poisson's ratios = 0·3, $(a_1/a_2)=1$, $(b_1/a_1)=\frac{1}{2}$

	$\frac{b_2}{b_1}=1$, Square fibers				$\frac{b_2}{b_1}=\frac{3}{2}$, Rectangular fibers			
Q	New quotient ν_N	Equation (4.7)† $\bar{\bar{\nu}}_R$	Upper bound $\bar{\bar{\nu}}^*_R$	Rayleigh quotient $\bar{\nu}_R$	New quotient ν_N	Equation (4.7)† $\bar{\bar{\nu}}_R$	Upper bound $\bar{\bar{\nu}}^*_R$	Rayleigh quotient $\bar{\nu}_R$
	0·124	0·129	0·129	0·214	0·110	0·115	0·115	0·246
	0·236	0·245	0·245	0·469	0·208	0·218	0·217	0·468
1·0	0·627	0·696	0·688	1·510	0·565	0·617	0·614	2·219
	0·825	0·852	0·850	2·292	0·764	0·792	0·791	2·994
	0·832	0·873	0·872	2·307	0·865	0·970	0·963	3·031
	0·241	0·252	0·251	0·433	0·208	0·220	0·220	0·463
	0·448	0·471	0·470	0·884	0·389	0·412	0·410	0·872
2·0	0·553	0·602	0·596	1·341	0·524	0·569	0·566	2·109
	0·852	0·881	0·879	2·203	0·778	0·814	0·813	2·786
	0·867	0·928	0·924	2·290	0·856	0·959	0·955	2·820
	$\frac{b_2}{b_1}=1$, Circular fibers				$\frac{b_2}{b_1}=2$, Elliptical fibers			
	0·134	0·140	0·140	0·178	0·107	0·113	0·112	0·196
	0·253	0·264	0·263	0·404	0·201	0·213	0·212	0·380
1·0	0·694	0·752	0·743	1·508	0·636	0·679	0·674	2·621
	0·839	0·870	0·869	1·696	0·736	0·780	0·780	2·637
	0·864	0·922	0·919	1·727	0·817	0·910	0·905	2·778
	0·262	0·274	0·274	0·360	0·200	0·213	0·212	0·371
	0·485	0·512	0·510	0·764	0·375	0·399	0·398	0·708
2·0	0·597	0·636	0·631	1·271	0·576	0·611	0·608	2·409
	0·875	0·904	0·904	1·664	0·762	0·810	0·809	2·528
	0·911	0·984	0·975	1·739	0·808	0·900	0·897	2·642

†For each value of Q, only the first eigenvalue calculated from (4.7) will in general be an upper bound for the corresponding exact eigenvalue.

$$\tilde{H}=\sum_{\substack{\alpha,\beta,\gamma,\delta=0\\ \alpha\neq\gamma,\beta\neq\delta}}^{\pm M'} S\frac{C_{1111}-1}{\bar{n}_1+\bar{n}_2C_{1111}}+\sum_{\substack{\alpha,\beta,\gamma=0\\ \alpha=\gamma,\beta\neq\delta}}^{\pm M'} S\frac{(C_{1111}-1)n_2}{\bar{n}_1+\bar{n}_2C_{1111}}B_2$$
$$+\sum_{\substack{\alpha,\beta,\gamma,\delta=0\\ \alpha\neq\gamma,\beta=\delta}}^{\pm M'} S\frac{(C_{1111}-1)}{\bar{n}_1+\bar{n}_2C_{1111}}B_3+\sum_{\substack{\alpha,\beta,\gamma,\delta=0\\ \alpha=\gamma,\beta=\delta}}^{\pm M'} S, \tag{6.23}$$

where

$$C_{1111}=C^{(2)}_{1111}/C^{(1)}_{1111},\qquad C_{1122}=C^{(2)}_{1122}/C^{(1)}_{1111},\qquad C_{1212}=C^{(2)}_{1212}/C^{(1)}_{1111},$$
$$S=\left\{(Q+2\pi\alpha)(Q+2\pi\gamma)+\frac{C_{1212}}{4}(2\pi\beta n_o)(2\pi\delta n_o)\right\}$$
$$\times\{S^{\alpha\beta}_{11}(Q+2\pi\alpha)+S^{\alpha\beta}_{12}(2\pi\beta n_o)\}\{S^{\gamma\delta *}_{11}(Q+2\pi\gamma)+S^{\gamma\delta *}_{12}(2\pi\delta n_o)\}$$
$$+\left\{C_{1122}(2\pi\beta n_o)(Q+2\pi\gamma)+\frac{C_{1212}}{4}(Q+2\pi\alpha)(2\pi\delta n_o)\right\}$$

Table 3. Fiber-reinforced composites. Lower and upper bounds for first three eigenfrequencies: $M' = 2$, $\theta = 1$, both Poisson's ratios = 0·3, $n_o = 1, (b_1/a_1) = \frac{1}{2}$

	$\frac{b_2}{b_1} = 1$, Square fiber						$\frac{b_2}{b_1} = \frac{3}{2}$, Rectangular fiber					
	$C^{(2)}_{1111}/C^{(1)}_{1111} = 4$			$C^{(2)}_{1111}/C^{(1)}_{1111} = 10$			$C^{(2)}_{1111}/C^{(1)}_{1111} = 4$			$C^{(2)}_{1111}/C^{(1)}_{1111} = 10$		
	Lower bound Equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$	Lower bound equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$	Lower bound equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$	Lower bound equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$
	0·45	0·45	0·47	0·33	0·34	0·36	0·43	0·44	0·45	0·30	0·32	0·35
1·0	0·85	0·86	0·89	0·64	0·65	0·70	0·82	0·82	0·86	0·59	0·60	0·66
	2·20	2·36	2·42	1·38	1·76	1·86	1·96	2·30	2·43	0·33	1·66	1·98
	0·89	0·91	0·93	0·65	0·68	0·72	0·85	0·86	0·90	0·59	0·63	0·68
2·0	1·47	1·70	1·75	0·58	1·28	1·37	1·41	1·62	1·69	0·58	1·18	1·27
	1·90	1·97	2·03	1·32	1·49	1·62	1·83	1·92	2·04	1·11	1·42	1·71

Table 4. Fiber-reinforced composites. Lower and upper bounds for first three eigenfrequencies: $M' = 2$, $\theta = 1$, both Poisson's ratios = 0·3, $n_o = 1$, $(b_1/a_1) = \frac{1}{2}$

	$\frac{b_2}{b_1} = 1$, Circular fibers						$\frac{b_2}{b_1} = 2$, Elliptical fibers					
	$C^{(2)}_{1111}/C^{(1)}_{1111} = 4$			$C^{(2)}_{1111}/C^{(1)}_{1111} = 10$			$C^{(2)}_{1111}/C^{(1)}_{1111} = 4$			$C^{(2)}_{1111}/C^{(1)}_{1111} = 10$		
	Lower bound equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$	Lower bound equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$	Lower bound equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$	Lower bound equation (4.20) $\nu^{(l)}$	Upper bound or new quotient ν_N	Rayleigh quotient $\bar{\nu}_R$
	0·46	0·46	0·47	0·35	0·36	0·38	0·42	0·43	0·45	0·29	0·31	0·34
1·0	0·87	0·87	0·90	0·67	0·68	0·72	0·80	0·81	0·84	0·57	0·59	0·63
	2·28	2·42	2·48	1·51	1·87	1·95	2·06	2·32	2·44	0·97	1·69	1·95
	0·92	0·93	0·95	0·69	0·72	0·75	0·84	0·85	0·88	0·58	0·61	0·63
2·0	1·58	1·74	1·77	0·86	1·35	1·41	1·44	1·60	1·65	0·80	1·15	1·22
	1·96	2·00	2·06	1·43	1·56	1·66	1·85	1·93	2·03	1·20	1·43	1·67

$$\times\{S_{12}^{\alpha\beta}(Q+2\pi\alpha)+S_{22}^{\alpha\beta}(2\pi\beta n_o)\}\{S_{11}^{\gamma\delta*}(Q+2\pi\gamma)+S_{12}^{\gamma\delta*}(2\pi\delta n_o)\}$$

$$+\left\{C_{1122}(Q+2\pi\alpha)(2\pi\delta n_o)+\frac{C_{1212}}{4}(Q+2\pi\gamma)(2\pi\beta n_o)\right\}$$

$$\times\{S_{11}^{\alpha\beta}(Q+2\pi\alpha)+S_{12}^{\alpha\beta}(2\pi\beta n_o)\}\{S_{12}^{\gamma\delta*}(Q+2\pi\gamma)+S_{22}^{\gamma\delta*}(2\pi\delta n_o)\}$$

$$+\left\{(2\pi\beta n_o)(2\pi\delta n_o)+\frac{C_{1212}}{4}(Q+2\pi\alpha)(Q+2\pi\gamma)\right\}$$

$$\times\{S_{12}^{\alpha\beta}(Q+2\pi\alpha)+S_{22}^{\alpha\beta}(2\pi\beta n_o)\}\{S_{12}^{\gamma\delta*}(Q+2\pi\gamma)+S_{22}^{\gamma\delta*}(2\pi\delta n_o)\}. \quad (6.24)$$

Inequality (4.20) then yields the desired lower bounds. This is illustrated in Table 3 for square and rectangular, and in Table 4 for circular and elliptical fibers. In both these tables, $\theta = 1$, $M' = 2$, $(a_1/a_2) = 1$, $(b_1/a_1) = (1/2)$, and both Poisson's ratios are equal to 0·3. Reasonably accurate lower bounds for only the first three eigenvalues can be obtained from inequality (4.20) when $M' = 2$, whereas accurate upper bounds for the first $(2M' + 1)^2$ eigenvalues are given by the new quotient. These bounds further suggest that the new quotient is indeed an effective computational tool for this class of problems.

It should be noted that when the elasticity tensor C_{ijkl} is constant but the mass-density ρ is variable, admitting discontinuities, and when the test functions (2.21) are used, then the new quotient reduces to the displacement Rayleigh quotient (2.8) and hence yields upper bounds (see Remark 8). These bounds are then much better than those obtained from the stress Rayleigh quotient (4.7) by a Rayleigh–Ritz procedure. This is illustrated in Table 5 for a layered composite.

Table 5. Layered composites. Comparison between the new quotient (2.11), the displacement Rayleigh quotient (2.8), and the stress Rayleigh quotient (4.7): $M' = 1$, $n_1 = n_2 = \frac{1}{2}$, and $\rho_2/\rho_1 = \theta$ and $\eta_2/\eta_1 = \gamma$ as indicated

	$\theta = 1,\ \gamma = 100$				$\theta = 100,\ \gamma = 1$			
Q	Exact	New quotient ν_N	Displacement Rayleigh quotient $\bar{\nu}_R$	Stress Rayleigh Quotient $\bar{\bar{\nu}}_R$	Exact	New quotient ν_N	Displacement Rayleigh quotient $\bar{\nu}_R$	Stress Rayleigh quotient $\bar{\bar{\nu}}_R$
	0·20	0·20	0·46	0·20	0·99	0·99	0·99	2·35
1·0	1·08	1·18	5·32	1·18	5·48	5·98	5·98	26·86
	1·79	2·74	7·31	2·74	9·05	13·83	13·83	36·92
	0·37	0·37	0·89	0·37	1·89	1·89	1·89	4·50
2·0	0·96	1·05	4·46	1·05	4·88	5·30	5·30	22·54
	1·87	2·98	8·38	2·98	9·46	15·05	15·05	42·32

In this table two cases are compared: (1) when $\theta = 1$, i.e. ρ is constant, but $\gamma = 100$; (2) when $\gamma = 1$, i.e. η is constant, but $\theta = 100$. In the first case the new quotient λ_N reduces to the stress Rayleigh quotient $\bar{\bar{\lambda}}_R$, equation (4.7), whereas in the second case λ_N reduces to the displacement Rayleigh quotient $\bar{\lambda}_R$, equation (2.8). In either case one obtains upper bounds. In general, however, the new quotient gives neither upper nor lower bounds.

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